DATE OF EXAM **Solution**

SUBJECT NAME: Advanced Functional Analysis - FINAL OR MIDTERM Exam: Back Paper 2014-15 - Semester ENTER I or II: M. Math IInd Yr

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1. *Let X be a topological vector space. A subset A of X is said to be bounded if for every neighborhood V of 0, there exists $s > 0$, such that $A \subset tV$ for all $t > s$. Suppose A, B are bounded subsets of X, show that $A \cup B$ and $A + B$ are bounded.*

   Solution: We have A, B are bounded subsets of X. As there always exists a basis $\mathcal{B}$ of balanced neighbourhoods of the origin in X. We have that for any $V \in \mathcal{B}$ there exists λ, μ such that $A \subset \lambda V,\ B \subset \mu V$.

$$A \cup B \subset \lambda V \cup \mu V \subseteq \max\{\lambda, \mu\} V.$$

$$A + B \subset \lambda V + \mu V \subseteq (\lambda + \mu) V.$$

   □

2. *Let Y be a normed linear space. Let D be a non-empty subset of Y. Show that D is bounded if and only if for every $f \in Y^*$ (Y^* is the space of bounded linear functionals on Y),*

$$\sup\{|f(a)| : a \in D\} < \infty.$$

   Solution: We can find the proof from the book "Functional Analysis" by B.V. Limaye. 9.3 Theorem, Page 142.

   □

3. *Let H be a Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $M = span\{e_n : n \in \mathbb{N}\}$. Consider the set $S = \{\frac{1}{n} e_n\} \cup \{0\}$. Show that S is compact. Show that the closed convex hull of S is not compact in M, but it is compact in H.*

   Solution:

   □

4. *Let A be a bounded operator on a Hilbert space. Show that*

$$||A||^2 = ||A^* A||.$$

   Solution: We can find the proof from the book "Notes on Functional Analysis" by Rajendra Bhatia. Page- 115, 9. Theorem.

   □

5. *Let A be a unital C^*-algebra. Show that a linear functional ϕ on A is positive if and only if $||\phi|| = \phi(1)$*

   Solution: We can find the proof from the book "Functional Analysis: Spectral Theory" by V.S. Sunder. Page-124, Proposition 3.4.11. □

6. *Let $\mathcal{C}$ be the C^*-algebra of 2×2 matrices. For a matrix $X = [x_{ij}]$, define $\phi(X) = \frac{1}{3}(x_{11} + 2x_{22})$. Show that ϕ is a state on $\mathcal{C}$. Describe the GNS triple (H, π, ξ) of ϕ. Compute the dimension of H.*

Solution: $\square$